Lebanese American University **Spring 2012**

Byblos

Calculus IV **Date:** Apr/03/2012
Test #1 **Duration:** 2h

Name: **ID:**

I. **(30 Points)** Let f be a function defined over $]-\pi\ ,\ \pi[$ defined as

$$f(x) = \begin{cases} -\pi - x & \text{if} \quad -\pi < x \le -\frac{\pi}{2}, \\ x & \text{if} \quad -\frac{\pi}{2} \le x < \frac{\pi}{2}, \\ \pi - x & \text{if} \quad \frac{\pi}{2} \le x < \pi. \end{cases}$$

a. Sketch the graphic representation of f and show if f is even or odd.

b. Prove that the Fourier series of f is

$$\frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\sin\left(\frac{n\pi}{2}\right)\sin(nx)}{n^2}$$

c. Deduce the values of the sums:

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}, \quad \sum_{n=1}^{\infty} \frac{1}{n^2}, \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}.$$

II. **(30 Points)** For $y \in \mathbb{R}$, consider the function $x = e^{-y^2}$.

a. Find a parametrization for f of the form $\mathbf{r}(t) = f(t)\mathbf{i} + t\mathbf{j}$ with $t \in \mathbb{R}$ and f being a function to be determined. (This parametrization is considered in the questions that follow).

b. Verify that $\mathbf{r}$ is smooth and sketch its graphic representation.

c. Find the unit tangent vector $\mathbf{T}$ at any point of $\mathbf{r}(t)$.

d. Prove that $\forall t \neq 0$ the principal unit normal $\mathbf{N}$ is never parallel to the x-axis.

e. Find the curvature at any point of $\mathbf{r}(t)$.

f. Prove that the curvature is maximum at the point $P(1, 0)$ and find the osculating circle at this point.

III. **(40 Points)** The hyperbola of equation $\frac{x^2}{4} - y^2 = 1$ has two branches. The first branch lies in the side $x \ge 0$ and the second in the side $x \le 0$. We denote by $\mathcal{H}$ the part that lies in the side $x \ge 0$.

a. Verify that a parametrization for $\mathcal{H}$ can be

$$\mathbf{r}(t) = 2\cosh(t)\mathbf{i} + \sinh(t)\mathbf{j}, \quad t \in \mathbb{R}.$$

Show that $\mathcal{H}$ is smooth and sketch its graphic representation. (The parametrization mentioned above is considered in the questions that follow).

b. Find the velocity vector $\mathbf{v}(t)$ and the unit tangent vector $\mathbf{T}$.

c. Find the curvature κ as a function of t.

d. Prove that κ is maximum at the points $(2, 0)$.

e. Prove that κ does not have a minimum. Give a geometric interpretation of this result.

f. Find the osculating circle at the point $Q(2, 0)$.

g. Generalize the result obtained in d. in order to find at which point of the **whole** hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, the curvature κ is maximum.

IV. **(10 Points)** Consider the planar curve given in polar coordinates by $\rho = \rho(\theta)$, $a < \theta < b$. Show that the curvature is given by the formula

$$\kappa(\theta) = \frac{|2(\rho')^2 - \rho\rho'' + \rho^2|}{[\rho^2 + (\rho')^2]^{\frac{3}{2}}}, \quad \text{where } \rho' = \frac{d\rho}{d\theta}, \text{ and } \rho'' = \frac{d^2\rho}{d\theta^2}.$$

Lebanese American University **Spring 2012**

Byblos

Calculus IV
Test #2

Date: May/14/2012
Duration: 2h

Name: **ID:**

I. **(15 Points)** Consider the surface (S) of equation $F(x, y, z) = z - \cos\left(\frac{xy}{2\pi}\right) = 0$ and the point $P_0(\pi, \pi, 0)$.

a. Verify that P_0 is on the surface (S).

b. Find the equation of the plane (P), tangent on the surface at the point P_0.

c. Find a parametric equation for the line (L), normal to the surface at the point P_0.

II. **(15 Points)** Consider the function $f(x, y) = \cos(xy)$.

a. Linearize $f(x, y)$ near the point $P(\frac{\pi}{2}, 1)$.

b. Find an upper bound for the magnitude of the error $E(x, y)$ over the rectangle

$$R: \quad \left|x - \frac{\pi}{2}\right| < 0.1, \quad |y - 1| < 0.1$$

III. **(15 Points)** Let (P) be the paraboloid of equation $z = x^2+y^2$ and (S) the sphere of equation $x^2+y^2+z^2 = 2$.

a. Prove that the intersection of (P) and (S) is the circle (C) of equation $x^2 + y^2 = 1$ that lies in the plane $z = 1$.

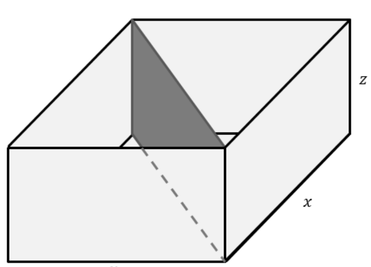
b. Let $\mathcal{R}$ be the region enclosed by (S) from above, (P) from below and lying in the first octant. Find the bounds of the region $\mathcal{R}$ using rectangular, cylindrical and spherical coordinates.

c. Find the volume of $\mathcal{R}$.

IV. **(15 Points)** Find the absolute maximum and minimum of the function $f(x, y) = -\frac{3}{2}x^2+xy+\frac{3}{2}y^2+x+3y+4$ over the triangle enclosed by the lines $y = x$, $y = -x$ and $y = -2$.

V. **(15 Points)** Let $x > 0$. Study the function $f(x, y) = x((\ln x)^2 + y^2)$ for local maxima, local minima and saddle points.

VI. **(15 Points)** We are going to manufacture a rectangular box with equal length and width, no top, one diagonal divider (see the figure below) and which has a fixed volume of 9 cm^3. It has metal divider, but cardboard sides. Metal costs $\sqrt{2}$ times as expensive as cardboard. For what dimensions x, and z is the cost minimized?



VII. **(15 Points)** Find the triple integral $\iiint_D xyz\, dx\, dy\, dz$ where D is the part of the sphere, centered at the origin and of radius 1, that lies in the first octant.

Lebanese American University **Spring 2012**

Byblos

Calculus IV **Date:** Jun/08/2012
Final Exam **Duration:** 2h

Name: **ID:**

I. **(15 Points)** Consider the vector field $\mathcal{F} = (-\sin(x+y)+2xe^{y+z})\,\mathbf{i}+(-\sin(x+y)+x^2e^{y+z})\,\mathbf{j}+(x^2e^{y+z})\,\mathbf{k}$.

a. Prove that $\mathcal{F}$ is conservative.
b. Find a potential function f, for the field $\mathcal{F}$.
c. Find the flow of $\mathcal{F}$ over the curve $\mathbf{r}(t) = \sin(t)\,\mathbf{i} + t\,\mathbf{j} + \sin(t)\,\mathbf{k}$ from $t = \pi$ to $t = 2\pi$.

II. **(15 Points)** Calculate the counterclockwise circulation of the vector field $\mathcal{F} = xy^3\,\mathbf{i} - y^3\,\mathbf{j}$ around the curve C which consists of the part of the parabola $y = x^2 - 1$ for $-1 \leq x \leq 1$ along with the positive semi-circle centered at the origin and joining $(-1, 0)$ to $(1, 0)$:

a. Using Green's theorem.
b. Directly using line integral.

III. **(15 Points)**

a. Find the volume of the solid enclosed by the paraboloid of equation $z = a^2 - x^2 - y^2$ from above and by the plane of equation $z = 0$ from below.
b. We denote by D the region inside the paraboloid $z = 5 - x^2 - y^2$ bounded below by the plane $z = 1$ and above by the plane $z = 4$. Find the outward flux of $\mathcal{F} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ across the boundary of D:
   i. Using the divergence theorem.
   ii. Directly using surface integral.

IV. **(15 Points)** Let (P) be the paraboloid of equation $x^2+y^2 = 2z$, and the vector field $\mathcal{F} = xy\,\mathbf{i}+xz^2\,\mathbf{j}+xy^2\,\mathbf{k}$. Let C be the intersection of (P) with the plan of equation $z = 2$.
Find the counterclockwise circulation of $\mathcal{F}$ around the curve C when viewed from above:

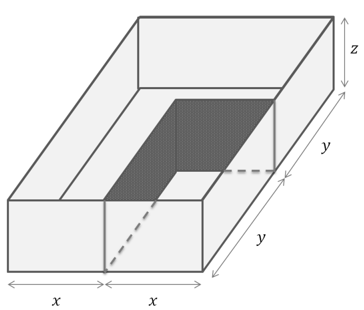
(a) Directly using line integral.
(b) Using Stokes' theorem in two different ways.

V. **(15 Points)** Study the function $f(x, y) = \dfrac{xy}{(1 + x^2)(1 + y^2)}$ for local maxima, local minima and saddle points.

VI. **(15 Points)** We are going to manufacture a rectangular box in order to pack an iPhone with its accessories. Apple suggests a box that has $2x$ as length, $2y$ as a width, z as a height, no top, two dividers (see the figure below) and a fixed volume of 72 cm^3. It has metal dividers, but cardboard sides. Metal costs 2 times as expensive as cardboard. For what dimensions Apple can minimize the cost of the box?

VII. **(15 Points)** Find the triple integral $\displaystyle\iiint_D \frac{dx\,dy\,dz}{\sqrt{x^2 + y^2 + z^2}}$,
where D is the domain limited by the two spheres

$$x^2 + y^2 + z^2 = 1 \text{ and } x^2 + y^2 + z^2 = 4.$$


z
y
y
x
x

Lebanese American University **Spring 2013**

Byblos

Calculus IV **Date:** Apr/03/2013
Test #1 **Duration:** 2h

Name: **ID:**

I. **(10 Points)** Short questions

a. Consider the curve $\mathcal{C}$ given by the vector function $\mathbf{r}(t) = \cos(t)\mathbf{i} + \sin(t)\mathbf{j} + \sin(t)\mathbf{k}$. Prove that $\mathcal{C}$ lies on **four** surfaces in the space and find their equations and types.

b. Consider the smooth curve $\mathcal{C}$ given by the polar equation $\rho = \rho(\theta)$. Prove that the arc length of the curve $\mathcal{C}$ between θ_1 and θ_2 (where $\theta_1 < \theta_2$) is given by the formula

$$L = \int_{\theta_1}^{\theta_2} \sqrt{(\rho'(\theta))^2 + (\rho(\theta))^2} d\theta.$$

II. **(35 Points)** Let f be a function defined over $]-\pi\ ,\ \pi[$ as

$$f(x) = \begin{cases} -x - \frac{\pi}{2} & \text{if } -\pi < x < -\frac{\pi}{2}, \\ x + \frac{\pi}{2} & \text{if } -\frac{\pi}{2} < x < 0, \\ -x + \frac{\pi}{2} & \text{if } 0 < x < \frac{\pi}{2}, \\ x - \frac{\pi}{2} & \text{if } \frac{\pi}{2} < x < \pi. \end{cases}$$

a. Sketch the graphic representation of f.

b. Prove that the Fourier series of f is

$$\frac{\pi}{4} + \frac{2}{\pi} \sum_{n=1}^{\infty} \left(\frac{1 + (-1)^n - 2\cos\left(\frac{n\pi}{2}\right)}{n^2} \right) \cos(nx)$$

c. Prove that the Fourier series can be simplified to

$$\frac{\pi}{4} + \frac{2}{\pi} \sum_{n=0}^{\infty} \frac{\cos(2(2n+1)x)}{(2n+1)^2}$$

d. Deduce the values of

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}, \quad \sum_{n=1}^{\infty} \frac{1}{n^2}, \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}.$$

III. **(25 Points)** Consider the vector function given by $\mathbf{r}(t) = (a\cos(t) + \sin t)\mathbf{i} + (\cos(t) - a\sin(t))\mathbf{j}$ and $\mathcal{C}$, $t \in \mathbb{R}$ its representative curve.

a. Find the velocity $\mathbf{v}(t)$, the speed $|\mathbf{v}(t)|$ and prove that $\mathcal{C}$ is smooth.

b. Find the tangent $\mathbf{T}$.

c. Find the curvature κ and deduce the type of $\mathcal{C}$.

d. Find the normal vector $\mathbf{N}$ without using $\frac{d\mathbf{T}}{dt}$.

e. Prove that $\mathbf{r}$ can be parameterized as $\mathbf{r}(\tau) = \alpha\cos(\tau)\mathbf{i} + \alpha\sin(\tau)\mathbf{j}$ where τ is a parameter to express in function of t and α a constant to be expressed in function of a.

IV. **(40 Points)** Consider the curve $\mathcal{C}$ given by its polar equation $\rho = e^{-k\theta}$, where $k > 0$ and $\theta \in \mathbb{R}$. This curve is called the *logarithmic spirale.*

a. Find a parametrization for $\mathcal{C}$ of the form $\mathbf{r}(\theta) = x(\theta)\mathbf{i} + y(\theta)\mathbf{j}$.

b. Find the velocity $\mathbf{v}(\theta)$ and the speed $|\mathbf{v}(\theta)|$ then deduce the smoothness of $\mathcal{C}$.

c. Find the tangent vector $\mathbf{T}$ and the principal unit normal $\mathbf{N}$.

d. Find the curvature κ.

e. Prove that the curve $\mathcal{C}$ has neither maximum nor minimum curvature.

f. Prove that, $\forall t$, the angle between the position vector and the tangent vector is constant.

g. Now consider the case where $k = 1$ and $\theta \in [0, +\infty)$

   i. Find the osculating circle at $\theta = \pi$.

   ii. Find the length of the whole curve.

Lebanese American University **Spring 2013**

Byblos

Calculus IV **Date:** May/09/2013
Test #2 **Duration:** 2h

Name: **ID:**

I. **(15 Points)** Consider the surface (S) of equation $F(x,y,z) = \cos\left(\frac{xy}{6z}\right) - \frac{\sqrt{3}}{2} = 0$ and the point $P_0(\pi, \pi, \pi)$.

a. Verify that P_0 is on the surface (S).
b. Find the equation of the plane (P), tangent on the surface at the point P_0.
c. Find a parametric equation for the line (L), normal to the surface at the point P_0.

II. **(15 Points)** Consider the function $f(x,y) = \cos(xy) + \sin(xy)$.

a. Linearize $f(x,y)$ near the point $P(\frac{\pi}{2}, 1)$.
b. Find an upper bound for the magnitude of the error $E(x,y)$ over the rectangle

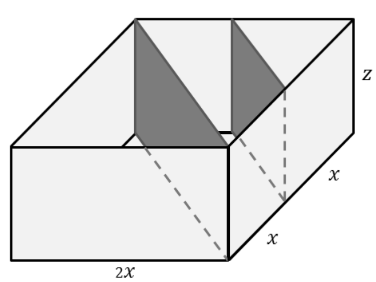
$$R: \quad \left|x - \frac{\pi}{2}\right| < 0.1, \quad |y-1| < 0.1$$

III. **(20 Points)** Let (S) be the sphere of equation $x^2 + y^2 + z^2 = R^2$. Consider the planes $(P_1): \ z = R/2$ and $(P_2): \ z = -R/2$.
Let $\mathcal{R}$ be the region enclosed by (S) laterally, (P_1) from above and (P_2) from below. Find the volume of $\mathcal{R}$ using cylindrical and spherical coordinates.

IV. **(15 Points)** Find the absolute maximum and minimum of the function $f(x,y) = x^3 + y^3 + 3xy$ over the domain $D = \{(x,y), |x| < 2, |y| < 2\}$.

V. **(15 Points)** Let $x > 0$. Study the function $f(x,y) = x^2((\ln x)^2 + x^2y^2)$ for local maxima, local minima and saddle points.

VI. **(20 Points)** We are going to manufacture a rectangular box with equal length and width, no top, two dividers (see the figure below) and which has a fixed volume of 98 cm^3. It has metal dividers, but cardboard sides. Metal costs $\sqrt{2}$ times as expensive as cardboard. What are the dimensions that minimize the cost of the box?



VII. **(10 Points)** Find the domain D such that the triple integral $\iiint_D (1 - 2x^2 - y^2 - z^2)\, dx\, dy\, dz$ is maximum.

Lebanese American University **Spring 2013**

Byblos

Calculus IV **Date:** Jun/01/2013
Final Exam **Duration:** 2h

Name: **ID:**

I. **(20 Points)** Consider the vector field $\mathcal{F} = (2xyz^2 - y\sin(xy))\,\mathbf{i} + (x^2z^2 - x\sin(xy))\,\mathbf{j} + (2x^2yz + e^z)\,\mathbf{k}$.

a. Prove that $\mathcal{F}$ is conservative.

b. Find a potential function f, for the field $\mathcal{F}$.

c. Find the flow of $\mathcal{F}$ over the curve $\mathbf{r}(t) = \sin(t)\,\mathbf{i} + \cos(t)\,\mathbf{j} + \sin(t)\,\mathbf{k}$ from $t = 0$ to $t = \frac{\pi}{2}$.

II. **(20 Points)** Calculate the counterclockwise circulation of the vector field $\mathcal{F} = xy^3\,\mathbf{i} - y^3\,\mathbf{j}$ around the curve C which consists of the part of the parabola $y = 1 - x^2$ for $-1 \leq x \leq 1$ along with the line connecting $(-1, 0)$ to $(0, -1)$ and the line connecting $(0, -1)$ to $(1, 0)$:

a. Using Green's theorem.

b. Directly using line integral.

III. **(30 Points)**

a. Prove that the volume of the solid enclosed by the cone of equation $z = \sqrt{\frac{x^2+y^2}{3}}$ from below and by the sphere of equation $x^2 + y^2 + z^2 = R^2$ from above is equal to $\frac{\pi}{3}R^3$.

b. Prove, using the surface integral that the area of the cap of the sphere $x^2 + y^2 + z^2 = R^2$ cut by the plane $z = \frac{R}{2}$ is equal to πR^2.

c. We denote by D the region inside the cone $z = \sqrt{\frac{x^2+y^2}{3}}$ bounded below by the sphere $x^2 + y^2 + z^2 = 1$ and above by the sphere $x^2 + y^2 + z^2 = 16$. Find the outward flux of $\mathcal{F} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ across the boundary of D:

   i. Using the divergence theorem.

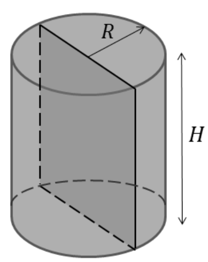
   ii. Directly using surface integral.

IV. **(15 Points)** Let C be the curve intersection of the plane $y + z = 2$ and the cylinder $x^2 + y^2 = 4$. Find the counterclockwise circulation of the vector field $\mathcal{F} = y\,\mathbf{i} + x^2\,\mathbf{j} + xz\,\mathbf{k}$ around the curve C when viewed from above, using Stokes' theorem **or** directly using a line integral.

V. **(20 Points)** We are going to manufacture a cylindrical box in order to pack a soft chocolate with biscuits. The manufacturer suggests a box that has R as radius, H as height with top, one divider (see the figure below) and a fixed volume of 27π cm^3. It has cardboard divider, but plastic sides. Cardboard costs π times as expensive as plastic. For what dimensions can the manufacturer minimize the cost of the box?

VI. **(10 Points)** Consider $a > 0$, $b > 0$ and $c > 0$.

Prove that the volume of the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$ is equal to $\dfrac{4}{3}\pi abc$.


R
H

Lebanese American University **Spring 2014**

Byblos

Calculus IV **Date:** Apr/07/2014
Test #1 **Duration:** 2h

Name: **ID:**

I. **(15 Points)** Short questions

a. What is the difference between the vector functions $r_1(t) = t\mathbf{i} + t^2\mathbf{j}$ and $r_2(t) = t^3\mathbf{i} + t^6\mathbf{j}$.

b. Consider the curve $\mathcal{C}$ given by the vector function $\mathbf{r}(t) = \cos(t)\mathbf{i} + \sin(t)\mathbf{k}$.
Prove that $\mathcal{C}$ lies, at least, on **three** surfaces in the space by giving their equations and types.

c. Suggest a parametrisation for a curve that has a non-zero constant curvature and a torsion equal to zero.

II. **(35 Points)** Let f be a function defined over $]-\pi\ ,\ \pi[$ as

$$f(x) = \begin{cases} (x+\pi)^2 & \text{if } -\pi < x < -\frac{\pi}{2}, \\ \dfrac{\pi^2}{4} & \text{if } -\frac{\pi}{2} < x < \frac{\pi}{2}, \\ (x-\pi)^2 & \text{if } \frac{\pi}{2} < x < \pi. \end{cases}$$

a. Sketch the graphic representation of f and show if it's even or odd.

b. Prove that the Fourier series of f is

$$\frac{\pi^2}{6} + \sum_{n=1}^{\infty} \left(\frac{2}{n^2} \cos\left(\frac{n\pi}{2}\right) + \frac{4}{\pi n^3} \sin\left(\frac{n\pi}{2}\right) \right) \cos(nx)$$

c. Prove that the Fourier series can be simplified to

$$\frac{\pi^2}{6} + \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} \cos\left((2n+1)x\right) + \frac{1}{2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos\left((2nx)\right)$$

d. Deduce the values of

$$\sum_{n=1}^{\infty} \frac{1}{n^2}, \quad \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3}, \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}.$$

III. **(40 Points)** Consider the curve $\mathcal{C}$ given by $\mathbf{r}(\theta) = a\theta\cos(\theta)\mathbf{i} + a\theta\sin(\theta)\mathbf{j}$, where $a > 0$ and $\theta \in \mathbb{R}$. This curve is called the *archimedian spirale*.

a. Find the velocity $\mathbf{v}(\theta)$ and the speed $|\mathbf{v}(\theta)|$ then deduce the smoothness of $\mathcal{C}$.

b. Find the tangent vector $\mathbf{T}$.

c. Find the curvature κ.

d. Prove that the curve $\mathcal{C}$ has a maximum curvature and find at which point it has this maximum curvature.

e. Show that the curve $\mathcal{C}$ has no minimum curvature.

f. Find the principal unit normal $\mathbf{N}$.

g. Find the osculating circle at the point of maximum curvature.

h. Show that $\forall\ \theta > 0$ the position vector is never orthogonal to the tangent.

IV. **(20 Points)** Consider $\mathbf{r}(t) : I \to \mathbb{R}^3$ such that the curve of $\mathbf{r}(t)$ is of torsion $\tau \neq 0$. We consider also that the curve lies on a sphere centered at the origin and of radius R and that it verifies $|r'(t)| = 1$. We call these curves *spherical curves*. We denote by κ its curvature, τ its torsion, $\mathbf{T}$ its unit tangent, $\mathbf{N}$ its principal unit normal and $\mathbf{B}$ its binormal.

a. Prove that $|\mathbf{r}(t)|^2 = R^2$ and that $\mathbf{r}(t) \perp \mathbf{T}$.

b. Deduce that $\mathbf{r} \cdot \mathbf{N} = -\dfrac{1}{\kappa}$. Deduce that $\exists\, a > 0$ such that $\forall t \in I$ we have $\kappa \geq a$.

c. Prove that $\mathbf{r} \cdot \mathbf{B} = \dfrac{\kappa'}{\kappa^2 \tau}$.

d. Deduce that the *spherical curves* verifies the relation

$$\left(\frac{1}{\kappa}\right)^2 + \left(\frac{\kappa'}{\kappa^2 \tau}\right)^2 = R^2$$

Lebanese American University **Spring 2014**

Byblos

Calculus IV **Date:** May/12
Test #2 **Duration:** 2h

Name: **ID:**

I. **(15 Points)** Consider the surface (S) of equation $F(x, y, z) = \sin\left(\frac{\pi x}{2z}\right) - e^{x^2-y^2} = 0$ and the point $P_0(1, 1, 1)$.

a. Verify that P_0 is on the surface (S).

b. Find the equation of the plane (P), tangent on the surface at the point P_0.

c. Find a parametric equation for the line (L), normal to the surface at the point P_0.

II. **(15 Points)** Consider the function $f(x, y) = x\cos\left(\frac{\pi}{2}y\right) + y\cos\left(\frac{\pi}{2}x\right)$.

a. Linearize $f(x, y)$ near the point $P(\frac{2}{3}, \frac{2}{3})$.

b. Find an upper bound for the magnitude of the error $E(x, y)$ over the rectangle

$$R : \quad \left|x - \frac{2}{3}\right| < 0.1, \quad \left|y - \frac{2}{3}\right| < 0.1$$

III. **(25 Points)** Let $\mathcal{P}$ be the paraboloid of equation $z = 2 - x^2 - y^2$ and $\mathcal{C}$ the cone of equation $z = \sqrt{x^2 + y^2}$.

a. Prove that the intersection of $\mathcal{P}$ and $\mathcal{C}$ is a circle and find its radius and center.

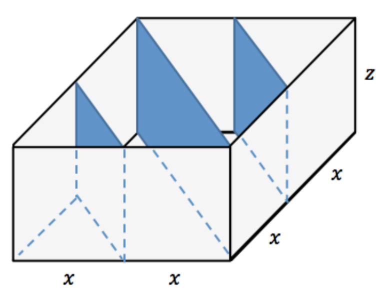
b. Let R be the ice cream limited from above by $\mathcal{P}$ and from below by $\mathcal{C}$. Define R in the rectangular, cylindrical and spherical systems of coordinates.

c. Find the volume of R using the cylindrical coordinates.

IV. **(15 Points)** Find the absolute maximum and minimum of the function $f(x, y) = x^3 - y^3 - 3xy$ over the domain $D = \{(x, y), |x| \leq 2, |y| \leq 2\}$.

V. **(20 Points)** Let $x > 0$. Study the function $f(x, y) = x^3((\ln x)^2 + 2x^2y^2)$ for local maxima, local minima and saddle points.

VI. **(20 Points)** We are going to manufacture a rectangular box with equal length and width, no top, three dividers (see the figure below) and which has a fixed volume of 128 cm^3. It has metal dividers, but cardboard sides. Metal costs $\sqrt{2}$ times as expensive as cardboard. What are the dimensions that minimize the cost of the box?



Lebanese American University **Spring 2014**

Byblos

Calculus IV **Date:** Jun-02
Final Exam **Duration:** 2h

Name: **ID:**

I. **(15 Points)** Consider the curve $\mathcal{C}$ of equation $\mathbf{r}(t) = \sin(3t)\,\mathbf{i} + \cos(t)\,\mathbf{j} + \cos(t)\,\mathbf{k}$ for $t = 0$ to $t = \frac{\pi}{3}$, and the vector field $\mathcal{F} = \left(\dfrac{1}{yz} + ye^{xy}\right)\mathbf{i} + \left(xe^{xy} - \dfrac{x}{y^2 z} - \pi z\sin(\pi yz)\right)\mathbf{j} - \left(\dfrac{x}{yz^2} + \pi y\sin(\pi yz)\right)\mathbf{k}$.
Find the flow of $\mathcal{F}$ over the curve $\mathcal{C}$.

II. **(15 Points)** Calculate the counterclockwise circulation of the vector field $\mathcal{F} = xy^3\,\mathbf{i} - x^3 y\,\mathbf{j}$ around the closed curve $\mathcal{C}$ formed by the parabolas $y = 1 - x^2$ and $y = 2 - 2x^2$:

a. Using Green's theorem.

b. Directly using line integral.

III. **(35 Points)** Let $a > 0$. Consider the paraboloid P of equation $z = a^2(x^2 + y^2)$ and the plan Q of equation $z = 1$.

a. Prove that the volume of the solid enclosed by P from below and Q from above is equal to $\dfrac{\pi}{2a^2}$.

b. Find the volume of the solid S enclosed laterally by the paraboloids P_1 of equation $z = x^2 + y^2$ and P_2 of equation $z = 3(x^2 + y^2)$ and from above by the plan $z = 1$ using:

   i. the result found in a.,

   ii. triple integrals with cylindrical coordinates,

   iii. triple integrals with spherical coordinates.

c. Prove, using the surface integral that the area of the paraboloid P situated below the plan Q is equal to

$$\frac{\pi}{6a^4}\left((4a^2+1)^{\frac{3}{2}} - 1\right).$$

d. We denote by D the boundary of the solid S defined in b. . Find the outward flux of $\mathcal{F} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ across the boundary of D:

   i. Using the divergence theorem.

   ii. Directly using surface integral.

IV. **(20 Points)** Consider the paraboloid P of equation $z = x^2 + y^2$ and the cone K of equation $z = \sqrt{x^2 + y^2}$.

a. Prove that, for $z > 0$, the intersection of P and K is a circle (denoted by C) and determine its center and radius.

b. Find the counterclockwise circulation of the vector field $\mathcal{F} = y\mathbf{i} - x\mathbf{j}$ around C when viewed from above using

   i. Stokes' theorem in three different ways.

   ii. line integral.

V. **(15 Points)** Study the function $f(x, y) = xy\ln(x)\ln(y)$ for local maxima, local minima and saddle points.

VI. **(10 Points)** Prove that the area of the lateral surface of the regular cone of revolution of height H and of circular basis of radius R is equal to

$$\pi R\sqrt{R^2 + H^2}.$$

Lebanese American University **Spring 2015**

Byblos

Calculus IV **Date:** Mar/17/2015
Test #1 **Duration:** 2h

Name: **ID:**

I. **(25 Points)** Short questions

a. What is the difference between the vector functions $r_1(t) = t\mathbf{i} + t^2\mathbf{j}$ and $r_2(t) = t^3\mathbf{i} + t^6\mathbf{j}$.

b. Consider the curve $\mathcal{C}$ given by the vector function $\mathbf{r}(t) = \cos(2t)\mathbf{i} + \cos(t)\mathbf{j} + \sin(t)\mathbf{k}$. Prove that $\mathcal{C}$ lies, at least, on **four quadratic** surfaces in the space by giving their equations and types.

c. Suggest a parametrisation for a curve that has a non-zero constant curvature and a torsion equal to zero.

d. Find the equation of the osculating circle of the curve given by $y = x^3 - 3x$ at the point $P(1, -2)$.

e. Find the torsion of the curve given by

$$\mathbf{r}(t) = (e^t \sin t - e^{2t} \cos t)\mathbf{i} + (2e^t \sin t + e^{2t} \cos t)\mathbf{j} + (e^t \sin t - 5e^{2t} \cos t)\mathbf{k}.$$

II. **(30 Points)** Let f be a function defined over $]-\pi, \pi[$ as

$$f(x) = \begin{cases} (x+\pi)^2 & \text{if } -\pi < x < -\frac{\pi}{2}, \\ \dfrac{\pi^2}{4} & \text{if } -\frac{\pi}{2} < x < \frac{\pi}{2}, \\ (x-\pi)^2 & \text{if } \frac{\pi}{2} < x < \pi. \end{cases}$$

a. Sketch the graphic representation of f and show if it's even or odd.

b. Prove that the Fourier series of f is

$$\frac{\pi^2}{6} + \sum_{n=1}^{\infty} \left(\frac{2}{n^2} \cos\left(\frac{n\pi}{2}\right) + \frac{4}{\pi n^3} \sin\left(\frac{n\pi}{2}\right) \right) \cos(nx)$$

c. Prove that the Fourier series can be simplified to

$$\frac{\pi^2}{6} + \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} \cos((2n+1)x) + \frac{1}{2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos((2nx))$$

d. Deduce the values of

$$\sum_{n=1}^{\infty} \frac{1}{n^2}, \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)^3}, \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}.$$

III. **(40 Points)** Consider the curve $\mathcal{C}$ given by $\mathbf{r}(t) = at\cos(t)\mathbf{i} + at\sin(t)\mathbf{j}$, where $a > 0$ and $t \in \mathbb{R}$. This curve is called the *archimedian spirale*.

a. Find the velocity $\mathbf{v}(t)$ and the speed $|\mathbf{v}(t)|$ then deduce the smoothness of $\mathcal{C}$.

b. Find the unit tangent vector $\mathbf{T}$.

c. Find the curvature κ.

d. Prove that the curve $\mathcal{C}$ has a maximum curvature and find at which point it has this maximum curvature.

e. Show that the curve $\mathcal{C}$ has no minimum curvature.

f. Find the principal unit normal $\mathbf{N}$.

g. Find the osculating circle at the point of maximum curvature.

h. Show that $\forall\, t > 0$ the position vector is never orthogonal to the tangent.

IV. **(15 Points)** Consider $\mathbf{r}(t) : I \to \mathbb{R}^3$ such that the curve of $\mathbf{r}(t)$ is of torsion $\tau \neq 0$. We consider also that the curve lies on a sphere centered at the origin and of radius R and that it verifies $|r'(t)| = 1$. We call these curves *spherical curves.* We denote by κ its curvature, τ its torsion, $\mathbf{T}$ its unit tangent, $\mathbf{N}$ its principal unit normal and $\mathbf{B}$ its binormal.

a. Prove that $|\mathbf{r}(t)|^2 = R^2$ and that $\mathbf{r}(t) \perp \mathbf{T}$.

b. Deduce that $\mathbf{r} \cdot \mathbf{N} = -\dfrac{1}{\kappa}$.

Deduce that $\exists\, a > 0$ such that $\forall t \in I$ we have $\kappa \geq a$ (that is κ has a minimum).

Lebanese American University **Spring 2015**

Byblos

Calculus IV **Date:** April 21
Test #2 **Duration:** 2h

Name: **ID:**

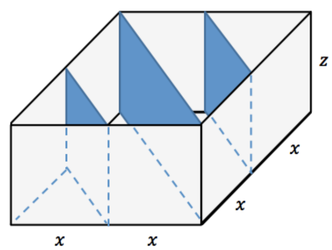
I. **(20 Points)** Consider the surface (S) of equation $F(x,y,z) = \dfrac{x+y}{z} - e^{x^2-y^2} = 0$ and the point $P_0(1,1,2)$.

  a. Verify that P_0 is on the surface (S).
  b. Find the equation of the plane (P), tangent on the surface at the point P_0.
  c. Find a parametric equation for the line (L), normal to the surface at the point P_0.

II. **(20 Points)** Find the absolute maximum and minimum of the function $f(x,y) = x^3 - y^3 - 3xy$ over the domain $D = \{(x,y), |x| \leq 2, |y| \leq 2\}$.

III. **(20 Points)** Let $x \in \mathbb{R}$. Study the function $f(x,y) = e^x(x^2 + e^{2x}y^2)$ for local maxima, local minima and saddle points.

IV. **(20 Points)** We are going to manufacture a rectangular box with equal length and width, no top, three dividers (see the figure below) and which has a fixed volume of 128 cm^3. It has metal dividers, but cardboard sides. Metal costs $\sqrt{2}$ times as expensive as cardboard. What are the dimensions that minimize the cost of the box?



V. **(10 Points)** Find the following triple integrals

  a. $\displaystyle\iiint_D (x+y)^2\, dxdydz$ where D is the cylinder of equation $x^2 + y^2 = 1$ with $0 \leq z \leq 1$,
  b. $\displaystyle\iiint_D \frac{e^z}{1+x+y+xy}\, dxdydz$ where D is the solid cube with $0 \leq x \leq 1$, $0 \leq y \leq 1$, $0 \leq z \leq 1$.

VI. **(20 Points)** Let $\mathcal{P}$ be the paraboloid of equation $z = 2 - x^2 - y^2$ and $\mathcal{C}$ the cone of equation $z = \sqrt{x^2+y^2}$.

  a. Prove that the intersection of $\mathcal{P}$ and $\mathcal{C}$ is a circle and find its radius and center.
  b. Let R be the ice cream limited from above by $\mathcal{P}$ and from below by $\mathcal{C}$. Define R in the rectangular and cylindrical systems of coordinates.
  c. Find the volume of R.

Lebanese American University **Spring 2015**

Byblos

Calculus IV **Date:** May-15
Final Exam **Duration:** 2h

Name: **ID:**

I. **(25 Points)** Consider the curve $\mathcal{C}$ of equation $\mathbf{r}(t) = \sin(3t)\,\mathbf{i} + \cos(t)\,\mathbf{j} + \cos(t)\,\mathbf{k}$ for $t = 0$ to $t = \frac{\pi}{3}$, and the vector field $\mathcal{F} = \left(\frac{a}{yz} + ye^{xy}\right)\mathbf{i} + \left(xe^{xy} - \frac{bx}{y^2 z} - \pi z \sin(\pi yz)\right)\mathbf{j} + \left(-\frac{x}{yz^2} - \pi y \sin(\pi yz)\right)\mathbf{k}$ where a and b are real numbers.

a. Find a and b such that $\mathcal{F}$ is conservative.
b. We consider a and b as found in a. . Find a potential function f for $\mathcal{F}$.
c. Deduce the flow of $\mathcal{F}$ over the curve $\mathcal{C}$.

II. **(20 Points)** Calculate the counterclockwise circulation of the vector field $\mathcal{F} = -xy^3\,\mathbf{i} + x^3 y\,\mathbf{j}$ around the closed curve $\mathcal{C}$ formed by the positive part of the circle of equation $x^2 + y^2 = 1$ and the line connecting the point $(-1, 0)$ to the point $(1, 0)$:

a. Using Green's theorem.
b. Directly using line integral.

III. **(35 Points)** Let $a > 0$. Consider the paraboloid P of equation $z = a^2(x^2 + y^2)$ and the plan Q of equation $z = 1$.

a. Prove that the volume of the solid enclosed by P from below and Q from above is equal to $\frac{\pi}{2a^2}$.
b. Find the volume of the solid S enclosed laterally by the paraboloids P_1 of equation $z = x^2 + y^2$ and P_2 of equation $z = 3(x^2 + y^2)$ and from above by the plan $z = 1$ using:
   i. the result found in part a.
   ii. triple integrals with spherical coordinates.
c. Prove, using the surface integral that the area of the paraboloid P situated below the plan Q is equal to

$$\frac{\pi}{6a^4}\left((4a^2 + 1)^{\frac{3}{2}} - 1\right).$$

d. We denote by D the boundary surface of the solid S defined in part b. . Find the outward flux of $\mathcal{F} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ across the boundary of D:
   i. Using the divergence theorem.
   ii. Directly using surface integral.

IV. **(20 Points)** Consider the paraboloid P of equation $z = x^2 + y^2$ and the cone K of equation $z = \sqrt{x^2 + y^2}$.

a. Prove that the intersection of P and K is a circle that we denote by C.
b. Find the counterclockwise circulation of the vector field $\mathcal{F} = y\mathbf{i} - x\mathbf{j}$ around C when viewed from above using
   a. Stokes' theorem in two different ways,
   b. line integral.

V. **(10 Points)** Prove, using surface integral, that the area of the lateral surface of the regular cone of revolution of height H and of circular basis of radius R is equal to

$$\pi R\sqrt{R^2 + H^2}.$$

Lebanese American University **Summer I 2015**

Byblos

Calculus IV **Date:** Jun/16/2015
Test #1 **Duration:** 2h

Name: **ID:**

I. **(35 Points)** Short questions

a. Compare the two vector functions $r_1(t) = t\mathbf{i} + t^2\mathbf{j}$ and $r_2(t) = t^3\mathbf{i} + t^6\mathbf{j}$.

b. Consider the curve $\mathcal{C}$ given by the vector function $\mathbf{r}(t) = \cos(2t)\mathbf{i} + \cos(t)\mathbf{j} + \sin(t)\mathbf{k}$. Prove that $\mathcal{C}$ lies, at least, on **four quadratic** surfaces in the space by giving their equations and types.

c. Suggest a parametrisation for a curve that has a non-zero constant curvature and a torsion equal to zero.

d. Find the equation of the osculating circle of the curve given by $y = x^3 - 3x$ at the point $P(1, -2)$.

e. Find the torsion of the curve given by

$$\mathbf{r}(t) = (e^t \sin t - e^{2t} \cos t)\mathbf{i} + (2e^t \sin t + e^{2t} \cos t)\mathbf{j} + (e^t \sin t - 5e^{2t} \cos t)\mathbf{k}.$$

II. **(25 Points)** Let f be a function defined over $]-\pi, \pi[$ as

$$f(x) = \begin{cases} -x - \pi/2 & \text{if } -\pi < x < -\frac{\pi}{2}, \\ 1 & \text{if } -\frac{\pi}{2} < x < \frac{\pi}{2}, \\ x - \pi/2 & \text{if } \frac{\pi}{2} < x < \pi. \end{cases}$$

a. Sketch the graphic representation of f.

b. Prove that the Fourier series of f is

$$\frac{1}{2} + \frac{\pi}{8} + \frac{2}{\pi} \sum_{n=1}^{\infty} \left(\frac{\sin(\frac{n\pi}{2})}{n} + \frac{(-1)^n - \cos(\frac{n\pi}{2})}{n^2} \right) \cos(nx)$$

c. Deduce the values of the following sums

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}, \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}.$$

III. **(35 Points)** Consider the curve $\mathcal{C}$ given by $\mathbf{r}(t) = at\cos(t)\mathbf{i} + at\sin(t)\mathbf{j}$, where $a > 0$ and $t \in \mathbb{R}$. This curve is called the *archimedian spirale*.

a. Find the velocity $\mathbf{v}(t)$ and the speed $|\mathbf{v}(t)|$ then deduce the smoothness of $\mathcal{C}$.

b. Find the unit tangent vector $\mathbf{T}$.

c. Find the curvature κ.

d. Prove that the curve $\mathcal{C}$ has a maximum curvature and find at which point it has this maximum curvature.

e. Show that the curve $\mathcal{C}$ has no minimum curvature.

f. Find the osculating circle at the point of maximum curvature.

g. Show that $\forall\, t > 0$ the position vector is never orthogonal to the tangent.

IV. **(15 Points)** Consider $\mathbf{r}(t) : I \to \mathbb{R}^3$ such that the curve of $\mathbf{r}(t)$ is of torsion $\tau \neq 0$. We consider also that the curve lies on a sphere centered at the origin and of radius R and that it verifies $|r'(t)| = 1$. We call these curves *spherical curves*. We denote by κ its curvature, τ its torsion, $\mathbf{T}$ its unit tangent, $\mathbf{N}$ its principal unit normal and $\mathbf{B}$ its binormal.

a. Prove that $|\mathbf{r}(t)|^2 = R^2$ and that $\mathbf{r}(t) \perp \mathbf{T}$.

b. Deduce that $\mathbf{r} \cdot \mathbf{N} = -\dfrac{1}{\kappa}$.

Deduce that $\exists\, a > 0$ such that $\forall t \in I$ we have $\kappa \geq a$ (that is κ has a minimum).

Lebanese American University **Summer I 2015**

Byblos

Calculus IV **Date:** July 02
Test #2 **Duration:** 2h

Name: **ID:**

I. **(20 Points)** Consider the surface (S) of equation $F(x,y,z) = \dfrac{x+y}{z} - e^{x^2-y^2} = 0$ and the point $P_0(1,1,2)$.

a. Verify that P_0 is on the surface (S).

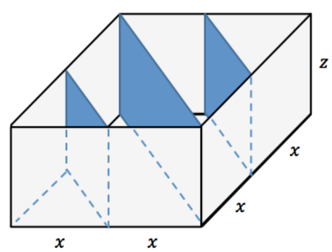
b. Find the equation of the plane (P), tangent on the surface at the point P_0.

c. Find a parametric equation for the line (L), normal to the surface at the point P_0.

II. **(20 Points)** Find the absolute maximum and minimum of the function $f(x,y) = x^3 - y^3 - 3xy$ over the domain $D = \{(x,y), |x| \leq 2, |y| \leq 2\}$.

III. **(15 Points)** Let $x \in \mathbb{R}$. Study the function $f(x,y) = e^x(x^2 + e^{2x}y^2)$ for local maxima, local minima and saddle points.

IV. **(20 Points)** We are going to manufacture a rectangular box with equal length and width, no top, three dividers (see the figure below) and which has a fixed volume of 128 cm^3. It has metal dividers, but cardboard sides. Metal costs $\sqrt{2}$ times as expensive as cardboard. What are the dimensions that minimize the cost of the box?



V. **(10 Points)** Find the following triple integrals

a. $\iiint_D (x+y)^2 \, dxdydz$ where D is the cylinder of equation $x^2 + y^2 = 1$ with $0 \leq z \leq 1$,

b. $\iiint_D \dfrac{e^z}{1+x+y+xy} \, dxdydz$ where D is the solid cube with $0 \leq x \leq 1$, $0 \leq y \leq 1$, $0 \leq z \leq 1$.

VI. **(25 Points)** Let $\mathcal{P}$ be the paraboloid of equation $z = 2 - x^2 - y^2$ and $\mathcal{C}$ the cone of equation $z = \sqrt{x^2+y^2}$.

a. Prove that the intersection of $\mathcal{P}$ and $\mathcal{C}$ is a circle and find its radius and center.

b. Let R be the ice cream limited from above by $\mathcal{P}$ and from below by $\mathcal{C}$. Define R using rectangular, cylindrical and spherical coordinates.

c. Find the volume of R.

Lebanese American University **Summer I - 2015**

Byblos

Calculus IV **Date:** July-10
Final Exam **Duration:** 2h

Name: **ID:**

I. **(25 Points)** Consider the curve $\mathcal{C}$ of equation $\mathbf{r}(t) = \sin(3t)\,\mathbf{i} + \cos(t)\,\mathbf{j} + \cos(t)\,\mathbf{k}$ for $t = 0$ to $t = \frac{\pi}{3}$, and the vector field $\mathcal{F} = \left(\frac{a}{yz} + ye^{xy}\right)\mathbf{i} + \left(xe^{xy} - \frac{bx}{y^2 z} - \pi z\sin(\pi yz)\right)\mathbf{j} + \left(-\frac{x}{yz^2} - \pi y\sin(\pi yz)\right)\mathbf{k}$ where a and b are real numbers.

a. Find a and b such that $\mathcal{F}$ is conservative.
b. We consider a and b as found in a. . Find a potential function f for $\mathcal{F}$.
c. Deduce the flow of $\mathcal{F}$ over the curve $\mathcal{C}$.

II. **(20 Points)** Calculate the counterclockwise circulation of the vector field $\mathcal{F} = -xy^3\,\mathbf{i} + x^3 y\,\mathbf{j}$ around the closed curve $\mathcal{C}$ formed by the negative part of the circle $x^2 + y^2 = 1$ and the line connecting the point $(-1, 0)$ to the point $(1, 0)$:

a. Using Green's theorem.
b. Directly using line integral.

III. **(35 Points)** Let $a > 0$. Consider the paraboloid P of equation $z = a^2(x^2 + y^2)$ and the plan Q of equation $z = 1$.

a. Prove that the volume of the solid enclosed by P from below and Q from above is equal to $\dfrac{\pi}{2a^2}$.
b. Find the volume of the solid S enclosed laterally by the paraboloids P_1 of equation $z = x^2 + y^2$ and P_2 of equation $z = 3(x^2 + y^2)$ and from above by the plan $z = 1$ using:
   i. the result found in part a.
   ii. triple integrals with spherical coordinates.
c. Prove, using surface integral that the area of the paraboloid P situated below the plan Q is equal to

$$\frac{\pi}{6a^4}\left((4a^2+1)^{\frac{3}{2}} - 1\right).$$

d. We denote by D the boundary surface of the solid S defined in part b. . Find the outward flux of $\mathcal{F} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ across the boundary of D:
   i. Using the divergence theorem.
   ii. Directly using surface integral.

IV. **(20 Points)** Consider the paraboloid P of equation $z = x^2 + y^2$ and the cone K of equation $z = \sqrt{x^2 + y^2}$.

a. Prove that the intersection of P and K is a circle that we denote by C.
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   a. Stokes' theorem in two different ways,
   b. line integral.

V. **(10 Points)** Prove, using surface integral, that the area of the lateral surface of the regular cone of revolution of height H and of circular basis of radius R is equal to

$$\pi R\sqrt{R^2 + H^2}.$$